

# A new description of the quantum superalgebra $U_q[sl(n+1|m)]$ and related Fock representations

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## Abstract

A description of the quantum superalgebra  $U_q[sl(n+1|m)]$  via creation and annihilation generators (CAGs) is given. A statement that the Fock representations of the CAGs provide microscopic realizations of exclusion statistics is formulated.

## 1 Description of $U_q[sl(n+1|m)]$ via deformed CAGs

First we introduce  $U_q[sl(n+1|m)]$  by means of its classical definition in terms of the Cartan matrix  $\alpha_{ij}$  and the Chevalley generators  $h_i, e_i, f_i$ ,  $i, j = 1, \dots, n+m$ . Let  $\mathbf{C}[[h]]$  be the complex algebra of formal power series in the indeterminate  $h$ ,

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$q = e^h \in \mathbf{C}[[h]]$ .  $U_q[sl(n+1|m)]$  is a Hopf algebra, which is a topologically free  $\mathbf{C}[[h]]$  module with generators  $h_i, e_i, f_i$ , subject to the Cartan-Kac relations ( $\bar{q} = q^{-1}$ )

$$[h_i, h_j] = 0, \quad (1a)$$

$$[h_i, e_j] = \alpha_{ij} e_j, \quad [h_i, f_j] = -\alpha_{ij} f_j, \quad (1b)$$

$$[[e_i, f_j]] = \delta_{ij} \frac{k_i - \bar{k}_i}{q - \bar{q}}, \quad k_i = q^{h_i}, \quad k_i^{-1} \equiv \bar{k}_i = q^{-h_i}, \quad (1c)$$

$$\alpha_{ij} = (1 + (-1)^{\theta_{i-1,i}}) \delta_{ij} - (-1)^{\theta_{i-1,i}} \delta_{i,j-1} - \delta_{i-1,j},$$

$$\theta_i = \begin{cases} 0, & \text{if } i = 0, 1, 2, \dots, n, \\ 1, & \text{if } i = n+1, n+2, \dots, n+m, \end{cases}; \quad \theta_{ij} = \theta_i + \theta_j,$$

the  $e$ -Serre relations

$$[e_i, e_j] = 0, \text{ if } |i - j| \neq 1; \quad e_{n+1}^2 = 0, \quad (2a)$$

$$[e_i, [e_i, e_{i\pm 1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i\pm 1}]_q]_{\bar{q}} = 0, \quad i \neq n+1, \quad (2b)$$

$$\begin{aligned} \{e_{n+1}, [[e_n, e_{n+1}]_q, e_{n+2}]_{\bar{q}}\} = \\ \{e_{n+1}, [[e_n, e_{n+1}]_{\bar{q}}, e_{n+2}]_q\} = 0, \end{aligned} \quad (2c)$$

and the  $f$ -Serre relations, obtained from the  $e$ -Serre relations by replacing everywhere  $e_i$  with  $f_i$ .

Here and everywhere:

$$\begin{aligned} [a, b]_x &= ab - xba, \quad \{a, b\}_x = ab + xba, \\ [[a, b]]_x &= ab - (-1)^{\deg(a)\deg(b)} xba. \end{aligned}$$

We do not write the other Hopf algebra maps  $(\Delta, \varepsilon, S)$ , since we will not use them. They are certainly also a part of the definition.

Introduce a normal order in the system of the positive roots [1,2]

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j | i < j = 0, \dots, n+m\}$$

as follows:

$$\varepsilon_i - \varepsilon_j < \varepsilon_k - \varepsilon_l \text{ if } j < l \text{ or if } j = l \text{ and } i < k. \quad (3)$$

Define the deformed CAGs to be Cartan-Weyl basis vectors, which are in agreement with the above normal order:

$$\begin{aligned} a_1^- &= e_1, \quad a_1^+ = f_1, \quad H_1 = h_1, \\ a_i^- &= [[[ \dots [ [e_1, e_2]_{\bar{q}_1}, e_3]_{\bar{q}_2}, \dots ]_{\bar{q}_{i-3}}, e_{i-1}]_{\bar{q}_{i-2}}, e_i]_{\bar{q}_{i-1}} = [a_{i-1}^-, e_i]_{\bar{q}_{i-1}}, \\ a_i^+ &= [f_i, [f_{i-1}, [\dots, [f_3, [f_2, f_1]_{q_1}]_{q_2} \dots ]_{q_{i-3}}]_{q_{i-2}}]_{q_{i-1}} = [f_i, a_{i-1}^+]_{q_{i-1}}, \\ H_i &= h_1 + (-1)^{\theta_1} h_2 + (-1)^{\theta_2} h_3 + \dots + (-1)^{\theta_{i-1}} h_i, \end{aligned} \quad (4)$$

where

$$q_i = q^{1-2\theta_i} = \begin{cases} q, & \text{if } i \leq n, \\ \bar{q}, & \text{if } i > n. \end{cases} \quad (5)$$

*Theorem.*  $U_q[sl(n+1|m)]$  is an unital associative algebra with generators  $H_i, a_i^\pm$ ,  $i = 1, \dots, n+m$  and relations

$$[H_i, H_j] = 0, \quad (6a)$$

$$[H_i, a_j^\pm] = \mp(1 + (-1)^{\theta_i} \delta_{ij}) a_j^\pm, \quad (6b)$$

$$[[a_i^-, a_i^+]] = \frac{L_i - \bar{L}_i}{q - \bar{q}}, \quad L_i = q^{H_i}, \quad \bar{L}_i \equiv L_i^{-1} = q^{-H_i} \quad (6c)$$

$$[[a_i^\eta, a_{i+\xi}^{-\eta}], a_k^\eta]_{q^{\xi(1+(-1)^{\theta_i} \delta_{ik})}} = \eta^{\theta_k} \delta_{k, i+\xi} L_k^{-\xi \eta} a_i^\eta, \quad (6d)$$

$$[[a_1^\xi, a_2^\xi]]_q = 0, \quad [[a_1^\xi, a_1^\xi]] = 0, \quad \xi, \eta = \pm \text{ or } \pm 1. \quad (6e)$$

## 2 Fock representations

The irreducible Fock representations are labelled by one non-negative integer  $p = 1, 2, \dots$ , called an order of the statistics. To construct them assume that the corresponding representation space  $W_p$  contains a cyclic vector  $|0\rangle$ , such that

$$a_i^- |0\rangle = 0, \quad H_i |0\rangle = p |0\rangle, \quad i = 1, 2, \dots, n+m; \quad (7)$$

$$[[a_i^-, a_j^+]] |0\rangle = 0, \quad i \neq j = 1, 2, \dots, n+m.$$

From (6) one derives that the deformed creation (resp. annihilation) generators  $q$ -supercommute,

$$[[a_i^\xi, a_j^\xi]]_q = 0, \quad i < j = 1, \dots, n+m, \quad \xi = \pm. \quad (8)$$

This makes evident the basis (or at least one possible basis) in a given Fock space, since any product of only creation generators can be always ordered.

As a basis in the Fock space  $W_p$  take the vectors

$$|p; r_1, r_2, \dots, r_{n+m}\rangle = \sqrt{\frac{[p - \sum_{l=1}^{n+m} r_l]!}{[p]! [r_1]! \dots [r_{n+m}]!}} (a_1^+)^{r_1} (a_2^+)^{r_2} \dots (a_n^+)^{r_n} \times$$

$$(a_{n+1}^+)^{r_{n+1}} (a_{n+2}^+)^{r_{n+2}} \dots (a_{n+m}^+)^{r_{n+m}} |0\rangle, \quad [x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad q \in \mathbf{R} \quad (9)$$

with

$$r_i \in \mathbf{Z}_+, \quad i = 1, \dots, n; \quad r_i \in \{0, 1\}, \quad i = n+1, \dots, n+m, \quad \sum_{i=1}^{n+m} r_i \leq p. \quad (10)$$

In order to write down the transformations of the basis under the action of the CAG's one has to write down the supercommutation relations between all Cartan-Weyl generators, expressed via the CAGs. The latter is a necessary condition for the application of the Poincare-Birghoff-Witt theorem, when computing the action of the generators on the Fock basis vectors. Bellow we display only some necessary relations, which follow from (6) in a rather nontrivial way:

$$L_i \bar{L}_i = \bar{L}_i L_i = 1, \quad L_i L_j = L_j L_i, \quad L_i a_j^\pm = q^{\mp(1+(-1)^{\theta_i \delta_{ij}})} a_j^\pm L_i, \quad (11)$$

$$\llbracket a_i^-, a_i^+ \rrbracket = \frac{L_i - \bar{L}_i}{q - \bar{q}}, \quad \llbracket a_i^\eta, a_j^\eta \rrbracket_q = 0, \quad \eta = \pm, \quad i < j, \quad (12)$$

$$\begin{aligned} \llbracket \llbracket a_i^\eta, a_j^{-\eta} \rrbracket, a_k^\eta \rrbracket_{q^{\xi(1+(-1)^{\theta_i \delta_{ik}})}} &= \eta^{\theta_j} \delta_{jk} L_k^{-\xi \eta} a_i^\eta + (-1)^{\theta_k} \epsilon(j, k, i) (q - \bar{q}) \llbracket a_k^\eta, a_j^{-\eta} \rrbracket a_i^\eta = \\ &= \eta^{\theta_j} \delta_{jk} L_k^{-\xi \eta} a_i^\eta + (-1)^{\theta_k \theta_j} \epsilon(j, k, i) q^\xi (q - \bar{q}) a_i^\eta \llbracket a_k^\eta, a_j^{-\eta} \rrbracket, \quad \xi(j - i) > 0, \quad \xi, \eta = \pm \end{aligned} \quad (13)$$

where

$$\epsilon(j, k, i) = \begin{cases} 1, & \text{if } j > k > i; \\ -1, & \text{if } j < k < i; \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

*Proposition.* The set of all vectors (9) constitute an orthonormal basis in  $W_p$  with respect to the scalar product, defined in the usual way with "bra" and "ket" vectors and  $\langle 0|0 \rangle = 1$ . The transformation of the basis under the action of the CAOs read:

$$H_i |p; r_1, r_2, \dots, r_{n+m}\rangle = \left( p - (-1)^{\theta_i} r_i - \sum_{j=1}^{n+m} r_j \right) |p; r_1, r_2, \dots, r_{n+m}\rangle, \quad (15)$$

$$\begin{aligned} a_i^- |p; r_1, \dots, r_{n+m}\rangle &= (-1)^{\theta_i(\theta_1 r_1 + \dots + \theta_{i-1} r_{i-1})} q^{r_1 + \dots + r_{i-1}} \sqrt{[r_i][p - \sum_{l=1}^{n+m} r_l + 1]} \\ &\times |p; r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_{n+m}\rangle, \end{aligned} \quad (16)$$

$$\begin{aligned} a_i^+ |p; r_1, \dots, r_{n+m}\rangle &= (-1)^{\theta_i(\theta_1 r_1 + \dots + \theta_{i-1} r_{i-1})} \bar{q}^{r_1 + \dots + r_{i-1}} (1 - \theta_i r_i) \sqrt{[r_i + 1][p - \sum_{l=1}^{n+m} r_l]} \\ &\times |p; r_1, \dots, r_{i-1}, r_i + 1, r_{i+1}, \dots, r_{n+m}\rangle. \end{aligned} \quad (17)$$

### 3 Properties of the underlying statistics

In the present talk we have defined the algebra  $U_q[sl(n+1|m)]$  in terms of new set of generators, called creation and annihilation generators. Let us illustrate

on a simple example of the nondeformed algebra  $sl(n+1|m)$  that within each Fock representation  $a_i^+$  (resp.  $a_i^-$ ) can be interpreted as operators creating (resp. annihilating) "particles" with, say, energy  $\varepsilon_i$ . Only for simplicity let us assume that  $n = m$ . Set moreover  $b_i^\pm = a_i^\pm$ ,  $f_i^\pm = a_{i+n}^\pm$ ,  $i = 1, \dots, n$ , and consider a "free" Hamiltonian  $H = \sum_{i=1}^n \varepsilon_i (H_i + H_{i+n}) = \sum_{i=1}^n \varepsilon_i (\llbracket b_i^+, b_i^- \rrbracket + \llbracket f_i^+, f_i^- \rrbracket)$ . Then  $[H, b_i^\pm] = \pm \varepsilon_i b_i^\pm$ ,  $[H, f_i^\pm] = \pm \varepsilon_i f_i^\pm$ . This result together with (nondeformed) Eqs. (16) and (17) allows one to interpret  $r_i$ ,  $i = 1, \dots, n$  as the number of  $b$ -particles with energy  $\varepsilon_i$  and similarly  $r_{i+n}$ ,  $i = 1, \dots, n$  as the number of  $f$ -particles with energy  $\varepsilon_i$ . Then  $b_i^+$  ( $f_i^+$ ) increases this number by one, it "creates" a particle in the one-particle state (= orbital)  $i$ . Similarly, the operator  $b_i^-$  ( $f_i^-$ ) diminishes this number by one, it "kills" a particle on the  $i$ -th orbital. On every orbital  $i$  there cannot be more than one particle of kind  $f$ , whereas such restriction does not hold for the  $b$ -particles. These are, so to speak, Fermi like (resp. Bose like) properties. There is however one essential difference. If the order of the statistics is  $p$  than no more than  $p$  "particles" can be accommodated in the system,  $\sum_{i=1}^{n+m} r_i \leq p$ . Hence the available places for new particles to be "loaded" on, say,  $i^{th}$  orbital depend on how many particles have been already accommodated on the other orbitals. This is neither Bose, nor Fermi like property. It is however a typical property for the so called *exclusion statistics* [3].

The statistics, which we have outlined above, is indeed an exclusion statistics. This statement will be proved rigorously elsewhere. The exclusion statistics reformulates the concept of fractional statistics as a *generalized Pauli exclusion principle* for spaces with arbitrary dimensions. Despite of the fact that exclusion statistics are defined for any space dimensions, so far they were applied and tested only within 1D and 2D models. The statistics described above are examples of microscopic description of exclusion statistics in arbitrary space dimensions.

## Acknowledgments

T.D.P. is grateful to Prof. H.D. Doebner for the invitation to attend the International Symposium on Quantum Theory and Symmetries, 18-22 July 1999, Goslar. N.I.S. is thankful to Prof. H.D. Doebner for the kind hospitality to work at ASI, TU Clausthal and to DAAD for the three months fellowship.

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